§10 Connectedness, Smoothness and Normality of Standard Abelian Covers.

Lemma 2 Assume Y connected and complete. Let $\pi : X \to Y$ be a standard abelian cover. Then X connected $\iff h^0(L_\chi^{-1}) = 0 \ \forall \chi \neq 1 \iff L_\chi \neq \mathcal{O}_Y \ \forall \chi \neq 1$.
In general, the number of connected components of X is $\#\{\chi \in G^* \mid L_\chi = \mathcal{O}_Y\}$.

proof Indeed, $\pi_* \mathcal{O}_X = \bigoplus_{\chi \in G^*} L_\chi^{-1}$, so X connected $\iff$

Y complete $\Rightarrow$ X complete because π is proper

$$1 = h^0(X, \mathcal{O}_X) = h^0(Y, \pi_* \mathcal{O}_X) = \sum_{\chi \in G^*} h^0(Y, L_\chi^{-1}) = \underbrace{h^0(Y, \mathcal{O}_Y)}_{\substack{\shortparallel \\ 1,\ Y \text{ is connected complete}}} + \sum{}' h^0(L_\chi^{-1})$$

$\iff h^0(L_\chi^{-1}) = 0 \quad \forall \chi \neq 1_G$.

Furthermore, $h^0(L_\chi^{-1}) = 0$ implies $L_\chi \neq \mathcal{O}_Y$.

It remains only to prove that if $L_\chi \neq \mathcal{O}_Y \ \forall \chi \neq 1_G$, then X is connected.

By contraddiction, assume X disconnected, so given a connected component A, then $\underset{\{g \in G \mid g\cdot A = A\}}{\underset{\shortparallel}{\mathrm{Stab}(A)}} \subsetneq G$ and X decomposes as

$$X = \bigsqcup_{g \in G/\mathrm{Stab}(A)} g \cdot A$$

Let us consider the quotient $A \xrightarrow{\pi_2} Y$

the group acting on A is $\mathrm{Stab}(A)$

Then $\pi_{2*} \mathcal{O}_A$ decomposes as a direct sum of eigensheaves $(\pi_{2*} \mathcal{O}_A)^\eta$, $\eta \in \mathrm{Stab}(A)^*$.

Each eigensheaf corresponds to some $\mathcal{L}_\chi$ of $\pi_* \mathcal{O}_X$; in particular $(\pi_{2*}\mathcal{O}_A)^{1_{G^*}}$ corresponds to some $\mathcal{L}_\eta^{-1}$, with $\eta \in G^*$ s.t. $\eta_{|\mathrm{Stab}(A)} = 1$. However, $A \xrightarrow{\pi_2} Y$ is a connected Galois cover, so $\mathcal{L}_\eta^{-1} = (\pi_{2*}\mathcal{O}_A)^{1_{G^*}} = \mathcal{O}_Y$
$\Rightarrow \mathcal{L}_\eta = \mathcal{O}_Y$ with $\eta \neq 1_{G^*}$; a contradiction. $\square$

Def π is TOTALLY RAMIFIED if

$$G' := \langle g \mid D_g \neq 0 \rangle = G$$

Warning: Someone uses totally ramified for "$\forall x \in X \ \ \mathrm{Stab}_G(x) = G$ or $\{1\}$".
The definitions are different but coincides if G is a cyclic group.

Lemma 3: π totally ramified $\Rightarrow X$ is connected.

proof: Choose g with $D_g \neq 0$. Let $p \in A$ be a point lying on D_g. Then g fixes p, so $g \cdot A \cap A \neq \emptyset \Rightarrow gA = A$.
This holds for any $g \in G$ with $D_g \neq 0$. Then $G' \cdot A = A$ but $G = G'$ since π is totally ramified, so $X = A$ is connected. $\square$

Remark Any standard abelian covers factorizes

$$\begin{tikzcd}
X \arrow[rr, "\pi"] \arrow[dr, "/G'"'] & & Y \\
& X/G' \arrow[ur, "/(G/G')"'] &
\end{tikzcd}$$

totally ramified $\qquad$ étale

Prop: If $\mathrm{Pic}(Y)$ has not torsion, then $|G/G'|$ is the number of connected components of X.

In particular, when $\mathrm{Pic}(Y)$ has not torsion, π is totally ramified $\Leftrightarrow$ X is connected

proof We use the previous factorization. Since $Z := X/G' \to Y$ is étale with group G/G', then by Pardini Equations $|X| \hat{L}_\chi \equiv 0 \quad \forall \chi \in (G/G')^*$ but $\mathrm{Pic}(Y)$ has not torsion $\Rightarrow \hat{L}_\chi \equiv 0 \ \forall \chi$. But then X/G' would be composed by exactly $|G/G'|$- conectected components by the previous Lemma 2. Thus, X has the same number $|G/G'|$ of conected components, and G' is acting on each of them. $\blacksquare$

Prop: Let $\pi: X \to Y$ be a standard abelian cover with building data $\{\mathcal{L}_\chi\}_{\chi \in G^*}$, $\{D_g\}_{g\in G}$.
Given a smooth point $x \in X$, then

(1) The tangent representation $\rho_x : \mathrm{Stab}(x) \to GL(T_{x,X})$ sending $g \in G$ to $dg_x : T_{x,X} \to T_{x,X}$ is faithful;

(2) $\mathrm{Stab}_G(x) = \bigoplus_{i=1}^{r} \langle g_i \rangle$ is the direct sum of the inertia groups $\langle g_i \rangle$ of the components T_{g_i} of $\mathrm{supp}(R)$ through x.

(3) T_{g_i} are smooth at x

(4) $T_{g_1}, \dots T_{g_r}$ are transversal at x

(5) $\pi(x)$ is smooth for Y.

(6) $\pi(T_{g_j})$ does occur with multiplicity 1 on D_{g_j} and does not occur on D_{g_i}, $i \neq j$.

(7) $\pi(T_{g_1}), \dots \pi(T_{g_r})$ are smooth and transversal at $\pi(x)$.

proof Let us consider $\mathrm{Fix}(g)$ and assume that it is not empty.
Then any connected component F of $\mathrm{Fix}(g) \cap X^0$ is a smooth irreducible algebraic variety with the same dimension equal to $m.g_{dg_x}(1)$ for the linear map dg_x, $x \in \mathrm{Fix}(g) \cap X^0$.
Indeed, given $x \in \mathrm{Fix}(g) \cap X^0$, so x is smooth for X,

then by Bochner-Cartan theorem there is a local chart $\mathcal{U}$ of x with coordinates $(x_1, \dots x_n)$ s.t.

$$(x_1, \dots x_n) \xrightarrow{g} dg_x \cdot (x_1, \dots x_n)$$

i.e. g is the multiplication map for its differential matrix at the point x

Thus, given $y \in \mathcal{U} \cap \mathrm{Fix}(g)$, then $y \overset{g}{\mapsto} y = dg_x \cdot y$

so $\mathcal{U} \cap \mathrm{Fix}(g) \cong \mathrm{Ker}(\mathrm{Id} - dg_x)$ which is a vector subspace, and so a smooth irreducible variety of dimension $m.g._{dg_x}(1)$.

We have proved
$$\mathrm{Fix}(g) \cap X^\circ \to \mathbb{N}$$
$$y \longmapsto m.g._{dg_y}(1)$$
is a locally constant continuous map, so it is constant on the connected components of $\mathrm{Fix}(g) \cap X^\circ$. However, G acts transitively on any connected comp. of $\mathrm{Fix}(g)$, so the dimension of any connect. component of $\mathrm{Fix}(g) \cap X^\circ$ is the same, and equal to $m.g._{dg_x}(1)$.

(1)

Let us consider now a smooth point x of X.

Then $\rho_x : \mathrm{Stab}_G(x) \to GL(T_{x,X})$ is faithful.

Indeed, assume $\rho_x(g) = \mathrm{Id}_{T_x X}$.

Then $d\rho_x = Id_{T_{x,X}}$, which implies

$$m_g \cdot d\rho_x(1) = \dim(X) = \dim(Y) \Rightarrow \mathrm{Fix}(g) \cap X^\circ$$

for a stand. ab. cover all conn. comp. of X have the same dim. eq. to $\dim(Y)$

has connect. components of dimension eq. to $\dim(Y)$. This is not possible if $g \neq 1_G$ as the locus of points of X fixed by g is a codimension 1 subvariety (is the locus lying on D_g). Thus, $g = 1_G$ and this prove ρ_x is faithful.

(2) Up to change a simultaneous diagonalizing basis of $T_{x,X}$, we have $\rho_x(g) = d\rho_x = \begin{pmatrix} \chi_1(g) & & 0 \\ & \ddots & \\ 0 & & \chi_n(g) \end{pmatrix}$, where

$\chi_i \in \mathrm{Stab}_G(x)^*$ are the charact of the repr. ρ_x.

Let $G_j := \mathrm{Ker}(\chi_1, \dots \underbrace{\hat{\chi}_j}_{\text{excluded}}, \dots \chi_n) \leq \mathrm{Stab}_G(x)$. Then G_j is a cyclic group. Indeed, $\chi_j : G_j \to \mathbb{C}^*$ is injective, because given $g \in G_j$ with $\chi_j(g) = 1$, then $\chi_i(g) = 1 \ \forall i$

$\Rightarrow \rho_x(g) = Id_{T_{x,X}} \underset{\rho_x \text{ is faithful}}{\Rightarrow} g = 1_G$.

Thus, $G_j = \langle g_j \rangle$, and $\mathrm{Stab}_G(x)$ decomposes as

$$\mathrm{Stab}_G(x) = \bigoplus_{i=1}^{n} \langle g_j \rangle$$

Indeed, consider $\underset{\langle g_1 \rangle}{h_1} + \dots + \underset{\langle g_n \rangle}{h_n} = 1_G$. Then

$$\underbrace{-h_1}_{\ker(\hat{\chi}_1,\chi_2,\dots\chi_n)\ni} = \underbrace{h_2+\dots+h_n}_{\in \ker(\chi_1)} \Rightarrow -h_1 \in \ker(\chi_1,\dots\chi_n) \Rightarrow \rho_x(-h_1) = Id_{T_x X}$$

$$\underset{\rho_x \text{ is faithful}}{\Rightarrow} -h_1 = 1_G \Rightarrow h_2+\dots+h_n = 1_G.$$

Recursively, we have $h_1 = \dots = h_n = 1_G$.

Consider $g \in Stab_G(x)$. Let $0 \le \alpha_i(g) \le |g_i| - 1$ be the unique integer s.t. $\chi_i(g) = e^{\frac{2\pi i}{|g_i|}\cdot \alpha_i(g)}$.

Then, $g - (\alpha_1(g)g_1 + \dots + \alpha_n(g)g_n) \in \ker(\chi_1,\dots\chi_n)$

$\underset{\rho_x \text{ faithful}}{\Rightarrow}$ it is equal to $1_G \Rightarrow g = \alpha_1(g)g_1 + \dots + \alpha_n(g)g_n$.

$\Rightarrow g \in \bigoplus_{i=1}^{n} \langle g_J \rangle$.

Let us consider those $g_1,\dots,g_r$ among $g_1,\dots g_n$ that are non-trivial, $g_J \neq 1_G$. Then by construct. of g_J, $\rho_x(g_J) = d_{g_J x} = \begin{pmatrix} 1 & & & & 0 \\ & \ddots & & & \\ & & \underset{\chi_J(g_J)}{\neq 1} & & \\ & & & \ddots & \\ 0 & & & & 1 \end{pmatrix}$, so we have

$m.g._{d_{g_J x}}(1) = n-1 \Rightarrow$ the connect. component T_{g_J} of $Fix(g_J) \cap X^\circ$ containing x is a smooth irreducible subvariety of dimension $n-1$ in X.

Furthermore, all the components T_g through x are are forced to be one among $T_{g_1},\dots T_{g_r}$.

This proves point (2).

(3) from point (2) we obtained $\mathrm{Stab}_G(x)$ acts as $(x_1, \dots x_n) \overset{g}{\longmapsto} dg_x(x_1, \dots x_n)$
$$= (\chi_1(g)x_1, \dots \chi_n(g)x_n)$$

In particular, we obtain $T_{g_J} = (x_J = 0)$, so they are smooth at x.

(4) $T_{g_1}, \dots T_{g_r}$ are transversal at x as they are the n-axis of the $(x_1, \dots x_n)$-chart.

(5) We notice that any g_J acts on U as a pseudo-reflection (namely it $g_J = \mathrm{diag}(1, \dots \chi_J(g_J), 1, \dots 1)$) and $\mathrm{Stab}_G(x)$ is gener. by pseudo-reflections. Thus, by Shephard-Todd theorem, then $\mathbb{C}[X]^{\mathrm{Stab}_G(x)}$ is a polinomial algebra generated by $x_1^{|\chi_1|}, \dots, x_n^{|\chi_n|}$ and $X/\mathrm{Stab}_G(x)$ is smooth at the image of x.
However, we consider the decomposition

$$\begin{tikzcd} X \arrow[rr, "\pi"] \arrow[dr, "\pi'"'] & & Y \\ & X/\mathrm{Stab}_G(x) \arrow[ur, "\text{étale locally at } \pi'(x)"'] & \end{tikzcd}$$

Thus a neigh. of $\pi(x)$ in Y is iso with a neigh. of $\pi'(x)$ in $X/\mathrm{Stab}_G(x)$ $\Rightarrow$ $\pi(x)$ is smooth.

(6) We consider the cover $X \xrightarrow{\pi'} X/\mathrm{Stab}_G(x)$.
So X is defined with the set of eq.

$$w_\chi \cdot w_\eta = w_{\chi\eta} \cdot \Bigg(\prod_{\substack{r_\chi^g + r_\eta^g \geq |g| \\ g \in \mathrm{Stab}_G(x)}} \sigma_g \Bigg)$$

We observe that $\chi_1, \dots \chi_r$ above generate $\mathrm{Stab}_G(x)^*$, and that $\forall \chi = \chi_1^{a_1} \cdots \chi_r^{a_r}$

$$\prod_{g \in \mathrm{Stab}_G(x)} \sigma_g^{q_\chi^g} \cdot w_\chi = \Big(\prod_{i=1}^{r} w_{\chi_i}^{a_i} \Big).$$

However, $q_\chi^{g_j} = \left\lfloor \dfrac{\sum a_i r_{\chi_i}^{g_j}}{|g_j|} \right\rfloor = 0$ by construction of π' (indeed $D_g \neq 0 \iff g \in \{g_1, \dots g_r\}$ and $\chi_i(g_j) = e^{\frac{2\pi i}{|g|} d_{ij}}$, so $q_\chi^{g_j} = 0$)

Thus, $\prod_{g \in \mathrm{Stab}_G(x)} \sigma_g^{q_\chi^g} \neq 0$ in a neigh. of x and so locally at x $\quad w_\chi = \Big(\prod w_{\chi_i}^{a_i}\Big)\Big(\prod \sigma_g^{q_\chi^g}\Big)^{-1}$, which is then a redundant equation.

This means X is defined locally at x by

$$w_{\chi_1}^{|\chi_1|} = \sigma_{g_1}, \ \dots, \ w_{\chi_r}^{|\chi_r|} = \sigma_{g_r}.$$

And x smooth forces $\sigma_{g_1}, \dots, \sigma_{g_r}$ to be reduced and with no common components

$$\mathrm{rk}\left(\begin{array}{c|c} \begin{matrix} \nabla\sigma_{g_1} \\ \vdots \\ \nabla\sigma_{g_r} \end{matrix} & 0 \\ \hline 0 & \begin{matrix} |\chi_1| w_{\chi_1}^{|\chi_1|-1} & & 0 \\ & \ddots & \\ 0 & & |\chi_r| w_{\chi_r}^{|\chi_r|-1} \end{matrix} \end{array}\right) = (n+r)-r = n \ \checkmark$$

This proves (6).

(7) $\pi(T_{g_J})$ are transversal at $\pi(x)$ (so, D_{g_J} are transversal at $\pi(x)$) because from (5) $\pi(T_{g_J})$ are the n-axis of the $(z_1, \dots z_n)$-chart, which are clearly transversal. $\boxed{/\!/}$

Proposition (Smoothness Property)

Let $\pi : X \to Y$ be a standard ab. cover with group G and building data $\{L_\chi\}_\chi, \{D_g\}_g$.

Then X is smooth over a point $y \in Y$ if and only if

(1) y is a smooth point of Y

(2) given $\Delta_1, \dots \Delta_r$ the irreducible components of $\mathrm{supp}(D)$ containing y, with $\Delta_J \leq D_{g_J}$, then Δ_J occurs with multiplicity 1 on D_{g_J} and it does not occur on D_{g_i}, $i \neq J$;

(3) Δ_J are smooth at y;

(4) $$\begin{aligned} \langle g_1 \rangle \oplus \dots \oplus \langle g_r \rangle &\longrightarrow G \quad \text{is injective,} \\ (h_1, \dots h_r) &\longmapsto \sum h_i \end{aligned}$$

(5) $\Delta_1, \dots, \Delta_r$ intersect transversally at y.

proof ($\Rightarrow$) x is smooth for X. By the previous proposition (5), $\pi(x) = y$ is smooth for Y. Instead, point (6) implies (2), point (7) implies point (3) and (5).
Finally, point (4) is obtained by point (2) of the previous proposition.
($\Leftarrow$) It is exactly the same proof of point (6) of the previous prop. $\boxed{\,}$

Corollary Y smooth.
Then X is normal $\Leftrightarrow$ $D = \sum_{g \in G} D_g$ is reduced (i.e. any prime divisors of Y occurs on D with multiplicity at most 1)

proof ($\Rightarrow$) Assume Δ occurs on D more than one time. Then $\Delta \le D_{g_1}$ and $\Delta \le D_{g_2}$ with $g_1 \neq g_2$, or $2\Delta \le D_g$.
In both cases, any point of X over Δ would be singular for X by point (2) of the previous proposition. Hence X is not smooth in codimension 1, which contradicts X is normal.
($\Leftarrow$) By the prev. prop, D reduce $\Rightarrow$ X smooth in

codimension 1. Furthermore the Serre property S_2 holds since $\pi_* \mathcal{O}_X$ is a free module on $\mathcal{O}_Y$ and Y satisfies S_2.

Remember:

X Noetherian Scheme is normal $\iff$

R_1 (Regular in Codimension 1): Sing(X) has codimension at least 2.

S_2: (Extension Property): Every Rational function defined on a op. set U whose complement has codimension ≥ 2 extends to the whole variety.

Corollary 2 X is smooth $\iff$

(1) Y is smooth

(2) $D = \sum_{g \in G} D_g$ is reduced, any $\Delta \leq D_g$ is smooth, $g \in G$, the intersections among the prime div. of D are transversal;

In other words, $D = \sum D_g$ is SIMPLE NORMAL CROSSING

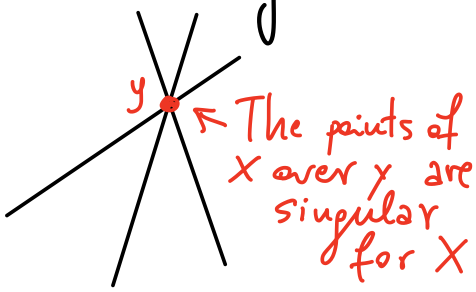
(3) $\forall x \in \Delta_1 \cap \cdots \cap \Delta_r$, with $\Delta_1 \leq D_{g_1}, \ldots \Delta_r \leq D_{g_r}$ then

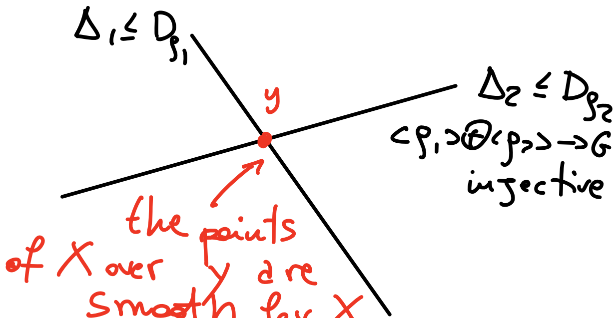
$\langle g_1 \rangle \oplus \cdots \oplus \langle g_r \rangle \longrightarrow G$ is injective.

Remark (The case of surfaces)

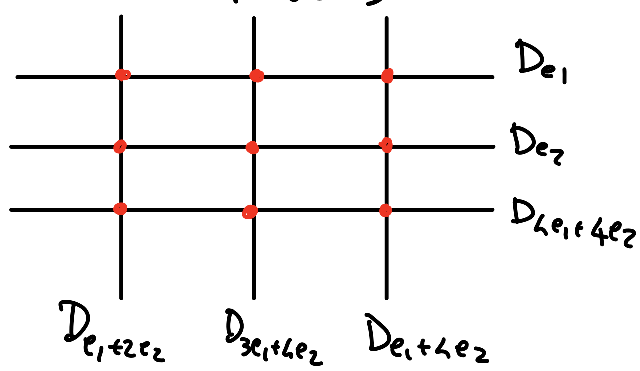
Smooth complex surfaces Y have dimension 2, so the number of codimension 1 subvarieties that can intersect transversally is at most 2.

Thus, for surfaces, $D = \sum_{g \in G} D_g$ is made by prime divisors that intersects two by two transversally, and no points of Y belong to more than two of them.





Example: The Beauville surface of Example 4 has a branch locus that is a grid in $\mathbb{P}^1 \times \mathbb{P}^1$:



Thus, D is simple normal crossing and the chosen labeling of the branch components verifies $\langle g_i \rangle \oplus \langle g_j \rangle \to G = \mathbb{Z}_5^2$ injective.

Thus, the Beauville surface $S \xrightarrow{\pi} \mathbb{P}^1 \times \mathbb{P}^1$ is smooth.